

Stable curves

We have ~~seen~~ encountered the Deligne-Mumford stack M_g of smooth curves of genus $g \geq 2$. There are a few shortcomings in considering M_g alone:

- (i) M_g is not proper over $\text{Spec } \mathbb{Z}$
- (ii) We need $g \geq 2$, $\bullet$

Our goal is to construct a proper smooth Deligne-Mumford stack $\overline{M}_g$ over $\text{Spec } \mathbb{Z}$, which can be seen as a "compactification" of M_g . For that we need to deviate a little from our world of smooth curves, and we need to take nodal curves also in our consideration. Moreover, we can construct a stack $\overline{M}_{g,n}$ of n -pointed genus g nodal curves. For the last stack $\overline{M}_{g,n}$, we can include $g = 0, 1$ too.

Assume $k = \bar{k}$ always, for simplicity.

Defⁿ: An n -pointed curve is a connected, nodal, projective curve C over a field k with an ordered collection of n ^{distinct} smooth points $P_1, P_2, \dots, P_n$. A point $q \in C$ is called special if C is either a node or one of the marked points $P_1, P_2, \dots, P_n$.

$(C, P_1, \dots, P_n)$ is called a stable curve if

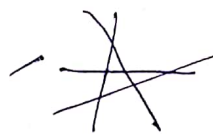
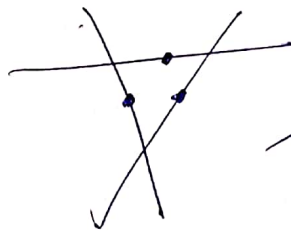
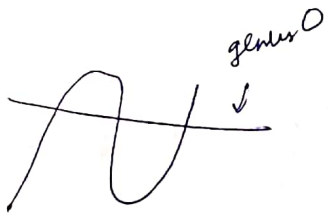
(1) Every smooth rational subcurve $|P| \subset C$ contains at least 3 special pts

(2) C is not genus 1 without marked pts.

(2)

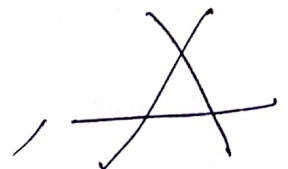
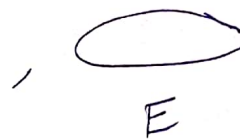
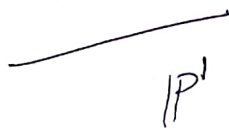
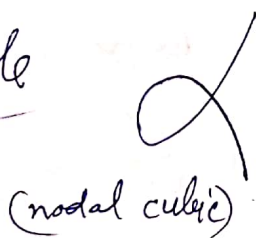
C is called semistable if ① is replaced by "every $\mathbb{P}^1 \subset C$ contains 2 special pts." we define C to be ^{pre}stable by dropping both ① and ②.

Eg:
stable;



Any genus ≥ 2 smooth curve

not stable



An automorphism of a ^{pre-}stable curve is an auto. $\alpha: C \xrightarrow{\sim} C$ with $\alpha(P_i) = P_i$.

Prop! Let $(C, P_1, \dots, P_n)$ be prestable. TFAE

- (1) $(C, P_1, \dots, P_n)$ is stable
- (2) $\text{Aut}(C, P_1, P_2, \dots, P_n)$ is finite
- (3) $\omega_C(P_1 + P_2 + \dots + P_n)$ is ample.

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Pf: Let $\tilde{C} \xrightarrow{\pi} C$ be the normalization. Let $\tilde{P}_1, \dots, \tilde{P}_n$ be the preimages of p_i under π , $\tilde{P}_1, \dots, \tilde{P}_m$ an ordering of the preimages of nodes. It is a straightforward exercise that $(C, p_1, \dots, p_n)$ is stable $\Leftrightarrow$ Each conn. component of $(\tilde{C}, \{\tilde{P}_i\}, \{\tilde{P}_j\})$ is stable.

(3)

For simplicity of notation, for a stable curve $(C, p_1, p_2, \dots, p_n)$, we define $\Omega_C = \omega_C(p_1 + p_2 + \dots + p_n)$. If $\tilde{C}_1, \tilde{C}_2, \dots, \tilde{C}_r$ are the conn. comp's of $\tilde{C}$, then it can be shown that $\pi^* \Omega_C|_{\tilde{C}_i} = \Omega_{\tilde{C}_i}$. So, Ω_C is ample $\Leftrightarrow$ each $\Omega_{\tilde{C}_i}$ is ample.

Finally, $\text{Aut}(C)$ is finite $\Leftrightarrow \text{Aut}(\tilde{C}_i)$ is finite $\forall i$, by the action of $\text{Aut}(C)$ on the finite set of marked points of $\bigcup_i \tilde{C}_i$.

So, we can WLOG assume C is smooth. Now the equivalences are straightforward. $\square$

Propⁿ: If C is a stable curve, then Ω_C^k is very ample $\forall k \geq 3$.

Pf: We show Ω_C^k separates pts and tangent vectors, i.e.,

$$H^0(C, \Omega_C^k) \rightarrow (\Omega_C^k \otimes k(x)) \oplus (\Omega_C^k \otimes k(y)),$$

$$H^0(C, \Omega_C^k) \rightarrow \Omega_C^k \otimes \frac{\mathcal{O}_{C, x}}{m_x^2} \quad \forall x, y \in C$$

It suffices to show $H^1(C, m_x m_y \otimes \Omega_C^k) = 0 \quad \forall x, y \in C$, not necessarily distinct.
 By Serre Duality, ETS ~~$H^0(C, m_x m_y \otimes \Omega_C^{l-k}) = 0$~~ $\text{Hom}_{\mathcal{O}_C}(m_x m_y, \Omega_C^{l-k}) = 0$, where
 M is the Weil divisor of marked pts.

A family of n -pointed stable curves is a flat, proper, (4)
 finitely presented morphism $\mathcal{C} \downarrow_S$ of schemes with n sections
 $\sigma_1, \dots, \sigma_n: S \rightarrow \mathcal{C}$ such that each geometric fiber
 $(\mathcal{C}_s, \sigma_1(s), \dots, \sigma_n(s))$ is a stable curve.

~~Prop.~~ In this case, $C \xrightarrow{\pi} S$ is an lci morphism, so we
 a relative dualizing line bundle $\omega_{C/S}$ that is compatible
 under base change, so on each geometric fibre C_s , $\omega_{C/S}$ restricts
 to the dualizing line bundle ω_{C_s} . Let $\Omega_{C/S}^i := \omega_{C/S}(\sum_i \sigma_i)$.

Propⁿ: If $k \geq 3$, $\Omega_{C/S}^k$ is rel. very ample over S , and $\pi_* \Omega_{C/S}^k$ is a vector bundle of rank $(2k-1)(g-1) + kn$.

Pf: Use previous proposition, and Grauert. $\square$

In particular, π is a projective morphism.

Openness of stability: If $\mathcal{C} \downarrow_S$ is a ~~max~~ family of nodal curves

$\sigma_1, \dots, \sigma_n$ n -sections, then the locus of pts $s \in S$ st $(\mathcal{C}_s, \{\sigma_i(s)\})$ is stable is open.

Pf: The locus in S st $\sigma_1(s), \dots, \sigma_n(s)$ are distinct, smooth is open.
so we can assume each $(\mathcal{C}_s, \{\sigma_i(s)\})$ is prestable. Now the locus of pts s such that $\Omega_{\mathcal{C}/S}|_{\mathcal{C}_s}$ is ample is open. $\square$

Deformation Theory

From now on, Ω refers to sheaf of
Kähler differentials (twisted by marked points) (3)

Theorem/Black Box: Let X_0 be a scheme of finite type over a field k such that X_0 is generically smooth and a local complete intersection. Let $A' \twoheadrightarrow A$ be a surjection of Noetherian local rings with residue field k . Assume that the kernel J satisfies $\cap_{A'} J = 0$ (in particular $J^2 = 0$). If $\begin{array}{c} X \\ \downarrow \\ \text{Spec } A \end{array}$ is a flat morphism of schemes with central fibre X_0 , then

(1) The group of automorphisms of a deformation $\begin{array}{c} X' \\ \downarrow \\ \text{Spec } A' \end{array}$ of $\begin{array}{c} X \\ \downarrow \\ \text{Spec } A \end{array}$ over A' is bijective to $\text{Ext}_{\mathcal{O}_{X_0}}^0(\Omega_{X_0}, J) = \text{Hom}_{\mathcal{O}_{X_0}}(\Omega_{X_0}, \mathcal{O}_{X_0}) \otimes_k J$.

(2) If there exists a deformation of $\begin{array}{c} X \\ \downarrow \\ \text{Spec } A \end{array}$ over A' , then the

set of isom. classes of all such deformations is bijective to

$$\text{Ext}_{\mathcal{O}_{X_0}}^1(\Omega_{X_0}, J) = \text{Ext}_{\mathcal{O}_{X_0}}^1(\Omega_{X_0}, \mathcal{O}_{X_0}) \otimes_k J.$$

(3) There is an element $ob_X \in \text{Ext}_{\mathcal{O}_{X_0}}^2(\Omega_{X_0}, J) = \text{Ext}_{\mathcal{O}_{X_0}}^2(\Omega_{X_0}, \mathcal{O}_{X_0}) \otimes_k J$ with the property that there exists a deformation of $\begin{array}{c} X \\ \downarrow \\ \text{Spec } A \end{array}$ over A' iff $ob_X = 0$.

(6)

Propⁿ: Let C be a stable curve of genus g . Then

$$\dim_k \operatorname{Ext}_{\mathcal{O}_C}^i(\Omega_C, \mathcal{O}_C) = \begin{cases} 0, & \text{if } i=0,2 \\ 3g-3+n, & \text{if } i=1. \end{cases}$$

... and that of dimension $3g-3+n$.

"Corollary": $\overline{\mathcal{M}}_{g,n}$ is a smooth DM stack of dimension $3g-3+n$.

(1) We have a k -finite type group scheme $\underline{\text{Aut}}(C)$ representing the functor $T \mapsto \text{Aut}(C_{T/H})$. We have shown $\underline{\text{Aut}}(C)$ is a finite abstract group. The Lie alg of $\underline{\text{Aut}}(C)$ is $\text{Ext}_{\mathcal{O}_C}^0(\Omega_C, \mathcal{O}_C) = 0$.

(take $A' = k[\epsilon]/(\epsilon^2)$, $A = k$ in blackbox)

So, $\underline{\text{Aut}}(C)$ is finite and discrete. This will show $\overline{\mathcal{M}}_{g,n}$ is D-M.

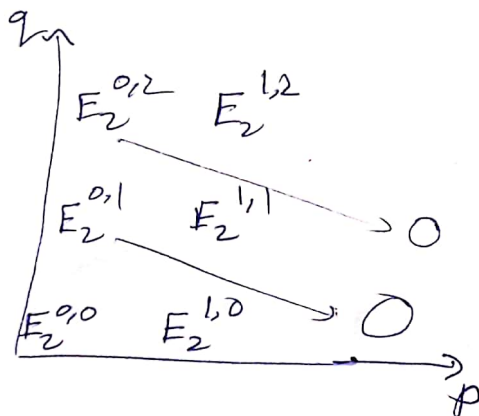
(2) The Zariski tangent space of $\overline{\mathcal{M}}_{g,n}$ at C is given by $\text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C)$ by blackbox, so $\overline{\mathcal{M}}_{g,n}$ has relative dimension $3g-3+n$ over $\text{Spec } \mathbb{Z}$.

(3) Smoothness of $\overline{\mathcal{M}}_{g,n}$ over $\text{Spec } \mathbb{Z}$ follows from vanishing of $\text{Ext}_{\mathcal{O}_C}^2(\Omega_C, \mathcal{O}_C)$ and by infinitesimal lifting criterion.

Pf of propn: For smooth C , the result follows easily by R-R and Ω_C is a line bundle. In the general case Ω_C has local complication, so we look at local Ext and relate it to global Ext by spectral sequence:

$$H^p(C, \text{Ext}_{\mathcal{O}_C}^q(\Omega_C, \mathcal{O}_C)) \Rightarrow \text{Ext}_{\mathcal{O}_C}^{p+q}(\Omega_C, \mathcal{O}_C).$$

Since $\dim C = 1$, $E_2^{p,q} = 0 \forall p \geq 2$.



We get an exact sequence:

$$0 \rightarrow E_2^{1,0} \rightarrow \text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C) \rightarrow E_2^{0,1} \rightarrow 0.$$

Ω_C is locally free away from the nodes, so $\text{Ext}_{\mathcal{O}_C}^1(\Omega_C, \mathcal{O}_C)$ is supported only at the nodes of C . So, $E_2^{1,1} = 0$,

$$E_2^{0,1} = \bigoplus_i \text{Ext}_{\mathcal{O}_{C,q_i}}^1(\Omega_{C,q_i}, \mathcal{O}_{C,q_i}) \quad (q_i\text{'s are the nodes})$$

$$= \bigoplus_i \text{Ext}_{\widehat{\mathcal{O}}_{C,q_i}}^1(\widehat{\Omega}_{C,q_i}, \widehat{\mathcal{O}}_{C,q_i})$$

$$= \bigoplus_i \text{Ext}_{\widehat{\mathcal{O}}_{C,q_i}}^1(\widehat{\Omega}_{C,q_i}, \widehat{\mathcal{O}}_{C,q_i})$$

$$E_2^{0,2} = \bigoplus_i \text{Ext}_{\hat{\mathcal{O}}_{C, q_i}}^2 (\Omega_{\hat{\mathcal{O}}_{C, q_i}}, \hat{\mathcal{O}}_{C, q_i}^*).$$

$$\hat{\mathcal{O}}_{C, q_i} = \frac{k[[x, y]]}{(xy) = 1}, \text{ we have locally free resolution } \Leftrightarrow$$

$$0 \rightarrow \hat{\mathcal{O}} \langle xy \rangle \xrightarrow{d} \hat{\mathcal{O}} \langle dx, dy \rangle \rightarrow \Omega_{\hat{\mathcal{O}}} \rightarrow 0$$

$$\parallel$$

$$\mathbb{I}/\mathbb{I}^2 \quad xy \rightarrow xdy + ydx$$

$$\text{ie, } 0 \rightarrow \hat{\mathcal{O}} \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} \hat{\mathcal{O}}^2 \rightarrow \Omega_{\hat{\mathcal{O}}} \rightarrow 0.$$

This computes $\text{Ext}_{\hat{\mathcal{O}}}^1(\Omega_{\hat{\mathcal{O}}}, \hat{\mathcal{O}}) = k$, $\text{Ext}_{\hat{\mathcal{O}}}^2(\Omega_{\hat{\mathcal{O}}}, \hat{\mathcal{O}}) = 0$.

So, $\text{Ext}^2 = 0$, $\dim E_2^{0,1} = 8 = \text{no. of nodes}$.

Now we compute $E_2^{1,0} = H^1(C, \text{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C))$.

Let $\tilde{C}$ be the ^{pointed} normalization of C , (~~$\tilde{\Sigma}$ the preimages of nodes~~)

One can show by local computations that

$$\text{Hom}_{\mathcal{O}_C}(\Omega_C, \mathcal{O}_C) = \pi_* \text{Hom}_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}, \mathcal{O}_{\tilde{C}}).$$

$$\therefore E_2^{1,0} = H^1(\tilde{C}, \text{Hom}_{\mathcal{O}_{\tilde{C}}}(\Omega_{\tilde{C}}, \mathcal{O}_{\tilde{C}})) \rightarrow \bigoplus_i$$

$$\dim E_2^{1,0} = \sum_i h^1(\tilde{C}_i, T_{\tilde{C}_i}(-\tilde{\Sigma}_i - \tilde{M}_i)) = \sum_i h^0(\tilde{C}_i, \omega_{\tilde{C}_i}^2(\tilde{\Sigma}_i + \tilde{M}_i))$$

$$= 3g - 3 - \delta$$

RR
+ genus
formula

$$\therefore \dim E_{\text{ext}}^1 = 3g - 3.$$

